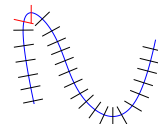


M421 HW 5



Due Friday Nov. 9

From Wade

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Non-book Exercises

1) For which $\alpha > 0$ is the function

$$f(x, y) = \begin{cases} \frac{x^2|y|^\alpha}{x^2 + |y|^3} & (x, y) \neq 0 \\ 0 & (x, y) = 0, \end{cases}$$

differentiable at zero?

2) Consider $R^\infty = \{\vec{x} = (x_1, x_2, x_3, \dots) \mid x_i \in \mathbf{R}, i = 1, 2, 3, \dots\}$. The l^2 norm on R^∞ is

$$\|\vec{x}\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

The space $l^2 = \{\vec{x} \in R^\infty \mid \|\vec{x}\|_2 < \infty\}$, is infinite dimensional. Show that the l^2 unit sphere, $S = \{\vec{x} \in l^2 \mid \|\vec{x}\|_2 = 1\}$, is closed, bounded, and **not** sequentially compact. That is, find a sequence from S which has no convergent subsequence.

Honor's Problems

3) Define the space $C^1[a, b] = \{f : [a, b] \mapsto \mathbf{R} \mid f \text{ and } f' \in C[a, b]\}$, and the norm

$$\|f\|_{1,1} = \int_a^b (|f(x)| + |f'(x)|) dx.$$

- (a) Show there exists $M > 0$ such that for all $f \in \mathcal{C}^1[a, b]$, $\|f\|_\infty \leq M\|f\|_{1,1}$.
Hint: Use the Fundamental Theorem of Calculus.
- (b) Show that if $\{f_n\}_{n=1}^\infty$ is a sequence from \mathcal{C}^1 and $f_n \rightarrow g$ in $\|\circ\|_{1,1}$ then $f_n \rightarrow g$ point wise.
- (c) Define $W^{1,1}$ to be the set of all sequences from \mathcal{C}^1 which are cauchy in the norm $\|\circ\|_{1,1}$. Show that for any sequence $\{f_n\}$ from \mathcal{C}^1 which is cauchy in $\|\circ\|_{1,1}$ there is a $g \in \mathcal{C}[a, b]$ such that

$$\|f_n - g\|_1 \rightarrow 0.$$

For this reason we say that

$$W^{1,1} \subset \mathcal{C}[a, b].$$

4)(a) Let $f \in L^1[a, b]$ and $g \in W^{1,1}[a, b]$. Show that the product $fg \in L^1[a, b]$. That is, if f is represented by the $\|\circ\|_1$ cauchy sequence $\{f_n\} \subset \mathcal{C}[a, b]$ and g by the $\|\circ\|_{1,1}$ cauchy sequence $\{g_n\} \subset \mathcal{C}^1[a, b]$, then the sequence $\{h_n\}$ where $h_n = f_n g_n$ is contained in $\mathcal{C}[a, b]$ and is cauchy in $\|\circ\|_1$.

(b) In part (a), show that if g is merely in $L^1[a, b]$, then the product fg may not be in $L^1[a, b]$. That is find two sequences $\{f_n\}$ and $\{g_n\}$, both from $\mathcal{C}[a, b]$ and both cauchy in $\|\circ\|_1$ such that the “product” $\{f_n g_n\}$ is not cauchy in $\|\circ\|_1$.