Math 942
HW1 – Due Monday Sept. 17

From Holmes

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1.5	23-25	1(aejp), 3, 5, 7
1.6	31-34	1(acf), 4, 5, 7

Non-book Exercises

1) Consider the perturbed polynomial

$$P_\epsilon(x) = P_0(x) + \epsilon g(x),$$

where P_0 has a zero of order k at $x = x_0$ and g has a zero of order $j < k$ at x_0. Show that P_ϵ has a zero of order j at x_0 and find the correction $x_\epsilon = x_0 + \epsilon^q x_1$ for the zeros which move away from x_0, (that is find q and an expression for x_1).

2) Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The matrix A has a two-dimensional eigenspace associated to the eigenvalue $\mu = 1$. Find the leading order correction (i.e. the two term expansion) for the two eigenvalues (and the associated eigenvectors) of the eigenspaces of $C = A + \epsilon B$, which perturb from $\mu = 1$ eigenspace of A.

3) Consider the eigenvalue problem

$$-u'' + \epsilon F(x) u = \lambda u \qquad 0 < x < 1,$$

$$u(0) = u(1), \qquad u'(0) = u'(1),$$

where F is a smooth, known function. Find a formula for the correction to the $n'th$ eigenvalue λ_n of the unperturbed problem. Hint: Show the problem is self-adjoint. With the periodic boundary conditions each unperturbed eigenvalue has two eigenfunctions, so λ_n may split upon perturbation!

1.5.1 Find a two-term asymptotic expansion of each soln of $x^2+x-\varepsilon=0$

Let $P_0(x) = x^2+x$

this has roots $x=0,\ x=-1$.

The polynomial

$P_\varepsilon(x) = x^2+x-\varepsilon$

has two roots $x_\varepsilon = x_0+\varepsilon x_1+\varepsilon^2 x^2 + O(\varepsilon^3)$

where $x_0 = 0$ or -1.

Plugging the expansion for x_ε into P_ε yields

$O(\varepsilon^0)$ $\quad x_0^2+x_0=0 \quad \Rightarrow \quad x_0 = 0, -1$

$O(\varepsilon^1)$ $\quad 2x_0x_1 + x_1 - 1 = 0 \quad \Rightarrow \quad x_1 = \dfrac{1}{2x_0+1}$

$O(\varepsilon^2)$ $\quad x_1^2 + 2x_0x_2 + x_2 = 0 \Rightarrow \quad x_2 = \dfrac{-x_1^2}{1+2x_0} = -\left(\dfrac{1}{1+2x_0}\right)^3$

$$\Rightarrow \quad x_\varepsilon = 0 + \frac{\varepsilon}{1} - \varepsilon^2 + O(\varepsilon^3)$$

$$\boxed{x_\varepsilon = \varepsilon - \varepsilon^2 + O(\varepsilon^3)}$$

or

$$x_\varepsilon = -1 + \frac{\varepsilon}{-1} - \frac{\varepsilon^2}{(-1)^3} + O(\varepsilon^3)$$

$$\boxed{x_\varepsilon = -1 - \varepsilon + O(\varepsilon^2)}$$

Two term expansions

1.5 #5

Find a two term expansion for the solns of

$$\tan\lambda = \lambda \quad \text{for} \quad \lambda \gg 1$$

To convert this to a contraction mapping we want a small derivative near the fixed point, $\tan\lambda$ has large derivatives $\Rightarrow$ $\tan^{-1}$ has small derivatives.

Solve

$$\lambda = \tan^{-1}\lambda + n\pi \qquad \text{where } \tan^{-1}\lambda \in [-\pi/2, \pi/2)$$

or $g(\lambda) \equiv \tan^{-1}\lambda + n\pi = \lambda$

For $\lambda \gg 1$ we expect roots for $\lambda \simeq \frac{2n+1}{2}\pi = (n+\frac{1}{2})\pi$

We take $\lambda_0 = (n+\frac{1}{2})\pi$

observe $g'(\lambda) = \frac{1}{1+\lambda^2}$

so $g'(\lambda_0) = O(\frac{1}{n^2})$

$$\lambda_1 = g(\lambda_0) = \tan^{-1}((n+\tfrac{1}{2})\pi) + n\pi$$

Satisfies $|\lambda_1 - \lambda_*| \le c\, g'(\lambda_0) \sim O(\frac{1}{n^2})$

How do we approximate $\tan^{-1}(x)$ for $x \gg 1$?

$$\tan^{-1}y - \tan^{-1}x = \int_x^y \frac{ds}{1+s^2}$$

$(t = s/x)$

so

$$\underbrace{\tan^{-1}\infty - \tan^{-1}x}_{= \frac{\pi}{2} - \tan^{-1}x} = \int_x^\infty \frac{ds}{1+s^2} = \frac{1}{x}\int_1^\infty \frac{dt}{\frac{1}{x^2}+t^2} = \frac{1}{x}\int_1^\infty \left(\frac{1}{t^2} - \frac{1}{x^2t^2} + O(\tfrac{1}{x^4})\right)$$

$$= \frac{1}{x} - \frac{1}{3x^3} + O\left(\frac{1}{x^5}\right)$$

1.5.5

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + O\left(\frac{1}{\lambda^5}\right)$$

We may reduce λ_1 to

$$\lambda_1 = \tan^{-1}\left[\left(n+\tfrac{1}{2}\right)\pi\right] + n\pi$$

$$= n\pi + \frac{\pi}{2} - \frac{1}{\left(n+\frac{1}{2}\right)\pi} + O\left(\frac{1}{n^2}\right)$$

$$\boxed{\lambda_1 = \left(n+\tfrac{1}{2}\right)\pi - \frac{1}{\left(n+\frac{1}{2}\right)\pi} + O\left(\frac{1}{n^2}\right)}$$

1.6.7 Find a 2- term expansion for

$$\begin{cases} u'''' - Ku'' + k^2 u = \varepsilon F_0 \sin \pi x & 0<x<1 \\ u''(0) = u''(1) = u(0) = u(1) = 0 \end{cases}$$

$k>0, \quad K \equiv \frac{1}{4}\int_0^1 (u')^2 dx \geq 0, \quad \varepsilon \ll 1$

First consider K as given and define

$$L_K u = u'''' - K u'' + k^2 u, \quad + \text{BC's}$$

Does L have a kernel?

$L_K \phi = 0$ then $(L_K \phi, \phi) = \int_0^1 \phi'''' \phi - K \int_0^1 \phi'' \phi \, dx + k^2 \int_0^1 \phi^2 dx$

$\xrightarrow{\text{use BC's}}$ $\quad = \int_0^1 (\phi'')^2 dx + K \int_0^1 (\phi')^2 dx + k^2 \int \phi^2 dx$

≥ 0

since $K>0$, $L_K \phi = 0 \Rightarrow \phi = 0$

Expand $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \Rightarrow K = K_0 + \varepsilon K_1 + \varepsilon^2 K_2$

At $O(\varepsilon^0)$

$$\begin{cases} u_0'''' - K_0 u_0'' + k^2 u_0 = 0 \\ K_0 = \int_0^1 (u_0')^2 dx \\ + \text{BCs} \end{cases} \Rightarrow L_{K_0} u_0 = 0$$

$\Rightarrow \boxed{u_0 = 0}$

$\boxed{K_0 = 0}$

1.6.7

At $O(\varepsilon)$ $\qquad K_1 = \frac{1}{2}\int_0^1 u_0' u_1' \, dx = 0$

$$\left[\begin{array}{l} L_0 u_1 \equiv u_1'''' + k^2 u_1 = F_0 \sin\pi x \\ u_1''(0/1) = u_1(0/1) = 0 \end{array}\right.$$

Since L_0 has no Kernel, this has a ! soln

Since $\quad L_0(\alpha \sin\pi x) = \alpha(\pi^4 + k^2)\sin\pi x \quad$ and $\left[\begin{array}{l}\sin(0) = \sin(\pi) = 0 \\ \sin''(\pi x) = \pi^2 \sin\pi x\end{array}\right.$

$$\Rightarrow \boxed{u_1 = \frac{F_0}{\pi^4 + k^2}\sin\pi x}$$

At $O(\varepsilon^2)$ $\qquad \boxed{K_2 = \frac{1}{4}\int_0^1 (u_1')^2 dx = \frac{1}{8}\frac{F_0^2}{(\pi^4+k^2)^2}}$

$$\left[\begin{array}{l} u_2'''' - (K_0 u_2'' + \underbrace{\cancel{K_2 u_0''}}_{0} + K_1 u_1'') + k^2 u_2 = 0 \\ \quad + BC \end{array}\right.$$

$$\left[\begin{array}{l} L_0 u_2 = 0 \\ +BC \end{array}\right. \quad \Rightarrow \quad u_2 = 0$$

At $O(\varepsilon^3)$

$$L_0 u_3 = \cancelto{0}{K_0} u_3'' + \cancelto{0}{K_1} u_2'' + K_2 u_1'' + \cancelto{0}{K_3 u_0''}$$

$$\left[\begin{array}{l} L_0 u_3 = K_2 u_1'' = -\dfrac{\pi^2 K_2 F_0}{\pi^4 + k^2}\sin\pi x \\ \quad +BC \end{array}\right.$$

As before

$$\boxed{u_3 = -\frac{\pi^2 K_2 F_0}{(\pi^4+k^2)^2}\sin\pi x = -\frac{\pi^2 F_0^3}{8(\pi^4+k^2)^4}\sin\pi x}$$

Two term expansion

$$\boxed{\begin{array}{l} u = \varepsilon u_1 + \varepsilon^3 u_3 \\ = \left(\varepsilon\dfrac{F_0}{\pi^4+k^2} - \varepsilon^3\dfrac{\pi^2 F_0^3}{8(\pi^4+k^2)^4}\right) \\ \qquad \times \sin\pi x \end{array}}$$

Non Book #2

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Find a 2-term expansion to the $\mu = 1$ eigenvalues

for $C = A + \varepsilon B$

With $\varepsilon = 0$

$C\vec{x} = \vec{x}$ has solns $\vec{x} = \vec{e}_1 = (1,0,0,0)$

$\vec{e}_2 = (0,1,0,0)$

A soln of

$$C\vec{x} = \lambda x$$

has

$$\vec{x} = \overbrace{\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2}^{\vec{x}_0} + \varepsilon\vec{x}_1$$

$$\lambda = 1 + \varepsilon\lambda_1$$

At $O(\varepsilon^0)$

$A\vec{x}_0 = \vec{x}_0$ which is satisfied

$O(\varepsilon^1)$

$$A\vec{x}_1 + B\vec{x}_0 = \vec{x}_1 + \lambda_1\vec{x}_0$$

(*) $\boxed{(A-1)\vec{x}_1 = (\lambda_1 - B)\vec{x}_0}$

this is soluable if RHS $\perp$ Kernel $(A-1) = \{\vec{e}_1, \vec{e}_2\}$

NB2)

$$\left(\left(\lambda_1 - B\right)\vec{x}_0 , \vec{e}_i\right) = 0 \qquad \text{for} \quad i = 1,2$$

$$\left(\lambda_1\left(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2\right) - \alpha_1 B\vec{e}_1 - \alpha_2 B\vec{e}_2 , \vec{e}_i\right) = 0 \qquad i = 1,2$$

$$\lambda_1\alpha_1\left(\vec{e}_1,\vec{e}_i\right) + \lambda_1\alpha_2\left(\vec{e}_2,e_i\right) - \alpha_1\left(B\vec{e}_1,\vec{e}_i\right) - \alpha_2\left(B\vec{e}_2,\vec{e}_i\right) = 0 \qquad i = 1,1$$

This is a 2x2 linear equation for α_1, α_2

$$\begin{bmatrix} \lambda_1 \overset{1}{(e_1,e_1)} - \overset{1}{(B\vec{e}_1,\vec{e}_1)} & \lambda_1 \overset{0}{(\vec{e}_2,\vec{e}_1)} - \overset{2}{(B\vec{e}_2,\vec{e}_1)} \\ \lambda_1 \overset{0}{(\vec{e}_1,\vec{e}_2)} - \overset{-1}{(B\vec{e}_1,e_2)} & \lambda_1 \overset{1}{(\vec{e}_2,\vec{e}_2)} - \overset{2}{(B\vec{e}_2,\vec{e}_2)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda_1 - 1 & -2 \\ -1 & \lambda_1 - 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0$$

Non-trivial solns require a zero determinant

$$\lambda_1 = 1 \Rightarrow (\alpha_1, \alpha_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_1 = 2 \Rightarrow (\alpha_1, \alpha_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{X}_{0,1} = (1,-1,0,0) + O(\varepsilon) \qquad \lambda = 1 + \varepsilon + O(\varepsilon^2)$$

eigenvectors

or

$$\vec{X}_{0,2} = (0,1,0,0) + O(\varepsilon) \qquad \lambda = 1 + 2\varepsilon + O(\varepsilon^2)$$

then from (*)

$$(A-1)\,\vec{x}_{1,1} = (1-B)\,x_{0,1}$$

$$(A-1)\,\vec{x}_{1,2} = (2-B)\,x_{0,2}$$

$$\begin{pmatrix} 0 & 0 & \vec{0} & \\ 0 & 0 & & \\ \vec{0} & & 2 & 1 \\ & & 1 & 2 \end{pmatrix} \vec{x}_{1,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Note Solvability

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} [illegible] \\ [illegible] \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x}_{1,1} = (0, 0, -1, 1)$$

$$\begin{pmatrix} 0 & 0 & \vec{0} & \\ 0 & 0 & & \\ \vec{0} & & 2 & 1 \\ & & 1 & 2 \end{pmatrix} \vec{x}_{1,2} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cancel{1/8} \\ \cancel{-3/8} \end{pmatrix} \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix}$$

$$\vec{x}_{1,2} = (0, 0, 1/8, -3/8)$$

$\lambda = 1 + \varepsilon$	$\vec{x}_1 = (1,-1,0,0) + \varepsilon\,(0,0,-1,1) + O(\varepsilon^2)$
$\lambda = 1 + 2\varepsilon$	$\vec{x}_2 = (0,1,0,0) + \varepsilon\,(0,0,+1/3,-2/3) + O(\varepsilon^2)$